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CHARACTERIZATION OF f-VECTORS OF FAMILIES OF CONVEX SETS IN $\mathbf{R}^d$ PART I: NECESSITY OF ECKHOFF'S CONDITIONS

BY
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ABSTRACT

Let $\mathcal{K}=\{K_1,\ldots,K_n\}$ be a family of n convex sets in $\mathbf{R}^d$. For $0\leqq i<n$ denote by f_i the number of subfamilies of $\mathcal{K}$ of size $i+1$ with non-empty intersection. The vector $f(\mathcal{K})=(f_0,f_1,\ldots)$ is called the f-vector of $\mathcal{K}$. In 1973 Eckhoff proposed a characterization of the set of f-vectors of finite families of convex sets in $\mathbf{R}^d$ by a system of inequalities. Here we prove the necessity of Eckhoff's inequalities. The proof uses exterior algebra techniques. We introduce a notion of generalized homology groups for simplicial complexes. These groups play a crucial role in the proof, and may be of some independent interest.

1. Introduction

Let $\mathcal{K}=\{K_1,K_2,\ldots,K_n\}$ be a family of sets. The *Nerve of* $\mathcal{K}$, $N(\mathcal{K})$, is an abstract simplicial complex on $N=\{1,2,\ldots,n\}$ defined by

$$N(\mathcal{K})=\left\{S\subset N: S=\varnothing \quad\text{or}\quad \bigcap_{i\in S}K_i\neq\varnothing\right\}. \tag{1.1}$$

In this paper we are concerned with families of convex sets in $\mathbf{R}^d$. We call a simplicial complex C *d-representable* if C is the nerve of a finite family of convex sets in $\mathbf{R}^d$. Recall that for a finite simplicial complex C, $f_i(C)$ is the number of i-dimensional faces of C, and the vector $f(C)=(f_0(C),f_1(C),\ldots)$ is called the f-vector of C. (For convenience, we regard $f(C)$ as an ultimately vanishing infinite sequence.) If $C=N(\mathcal{K})$, then $f_i(C)$ is the number of subfamilies of $\mathcal{K}$ of size $i+1$ with nonempty intersection.

The purpose of this paper is to characterize f-vectors of d-representable complexes. Such a characterization by means of a system of inequalities (see (1.7) below) was conjectured by Eckhoff in [5, 6] and was proved by him for the

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cases $d = 1, 2$. Eckhoff also proved the sufficiency part of his conjecture for all $d \geqq 1$, but this proof was not published. In this paper we prove the necessity of Eckhoff's inequalities, and thus establish Eckhoff's conjecture. In part II we will give an independent proof for the sufficiency part of the conjecture and will discuss some applications and related problems.

To introduce Eckhoff's inequalities we need some definitions and notations: For positive integers h and k, h can be written uniquely in the form

$$h = \binom{h_k}{k} + \binom{h_{k-1}}{k-1} + \cdots + \binom{h_j}{j}, \tag{1.2}$$

where $h_k > h_{k-1} > \cdots > h_j \geqq j \geqq 1$. This representation is called the *k-canonical representation* of h. Given this representation, define

$$h^{(k)} = \binom{h_k}{k-1} + \binom{h_{k-1}}{k-2} + \cdots + \binom{h_j}{j-1}. \tag{1.3}$$

For $h = 0$ put $0^{(k)} = 0$.

Canonical representations and the function $h^{(k)}$ were defined in the context of the following theorem, which characterizes f-vectors of arbitrary simplicial complexes.

KRUSKAL–KATONA THEOREM [13]. *An ultimately vanishing sequence $f = (f_0, f_1, \dots)$ of nonnegative integers is an f-vector of a simplicial complex iff*

$$f_k^{(k+1)} \leqq f_{k-1} \qquad \textit{for all } k \geqq 1. \tag{1.4}$$

For simple proofs of the Kruskal–Katona Theorem see [4] or [8]. Related results and an extensive bibliography may be found in [10].

Let C be a d-representable complex (d is a fixed positive integer). The h-vector of C, $h(C) = (h_0, h_1, \dots)$, is defined by

$$h_k = \begin{cases} f_k(C), & k = 0, 1, \dots, d-1, \\[2ex] \sum_{j \geq 0} (-1)^j \binom{k+j-d}{j} f_{k+j}(C), & k = d, d+1, \dots . \end{cases} \tag{1.5}$$

It is easy to verify that

$$f_k(C) = \begin{cases} h_k & k = 0, \dots, d-1, \\[2ex] \sum_{j \geq 0} \binom{k+j-d}{j} h_{k+j}, & k = d, d+1, \dots . \end{cases} \tag{1.6}$$

Now we are ready to formulate Eckhoff's conjecture.

ECKHOFF'S CONJECTURE [5, 6]. *$h = (h_0, h_1, \ldots)$ is the h-vector of a d-representable complex iff the following inequalities hold*:

$$\text{(1.7)} \qquad \begin{cases} \text{(i) } h_k \geqq 0, & k = 0, 1, 2, \ldots, \\ \text{(ii) } h_k^{(k+1)} \leqq h_{k-1}, & k = 1, \ldots, d-1, \\ \text{(iii) } h_k^{(d)} \leqq h_{k-1} - h_k, & k = d, d+1, \ldots . \end{cases}$$

In fact, Eckhoff's "h-Vermutung" ([6, p. 176]) asserts that the inequalities (1.7) characterize the h-vectors of d-representable complexes, of d-collapsible complexes (see below) and of two other classes of complexes. Remarks concerning these other classes appear at the end of this paper.

Our proof is based on multilinear techniques introduced in a previous paper [11]. However, the exposition in this paper does not depend essentially on [11]. In [11] we proved the following theorem, which was conjectured by Katchalski and Perles:

THEOREM. *For $d \geqq 1$, $k \geqq 0$, $n \geqq 0$ and $r \geqq 1$, the maximum of $f_{k-1}(C)$ over all d-representable complexes C with n vertices and* $\dim C < d + r$ *is* $\sum_{i=0}^{d} \binom{n-r}{i} \binom{r}{k-i}$. *(This maximum is attained, e.g., when C is the nerve of a family consisting of r copies of $\mathbf{R}^d$ and $n - r$ hyperplanes in general position in $\mathbf{R}^d$.)*

The Katchalski–Perles conjecture was proved independently by Eckhoff in [7].

In [20] Wegner proved that a d-representable complex is d-collapsible. We shall use this result and, in fact, we shall prove that inequalities (1.7) hold for every d-collapsible complex. The definition of d-collapsibility and related concepts are given in Section 2. The necessary algebraic background is given in Section 3. In Section 4 we give the basic definitions and results needed for the translation of our problem into the language of exterior algebra. The proof of the theorem is given in Sections 5 and 6. In Section 5 we define some kind of generalized homology groups for simplicial complexes and show that some of these groups vanish for d-collapsible complexes. In Section 6 we finally show how the vanishing of these groups implies Eckhoff's inequalities. The final section contains some concluding remarks.

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The following notations will be used in the paper: For a natural number m,

$[m]=\{1,2,\ldots,m\}$. For $j>i\geqq 0$, $[i,j]=\{i,i+1,\ldots,j\}$. For a set T and a nonnegative integer r, $T^{(r)}=\{S\subset T:|S|=r\}$. For a simplicial complex C and a nonnegative integer k,

$$\operatorname{skel}_k C=\{S\in C:\dim S\leqq k\}\qquad(=\{S\in C:|S|\leqq k+1\}),$$

$$C_k=\{S\in C:\dim S=k\}.$$

For a complex C and a face $S\in C$, the quotient complex C/S (known also as the link of S in C) is defined by

$$C/S=\{T\setminus S:T\in C,\ T\supset S\}. \tag{1.8}$$

2. d-collapsibility of simplicial complexes

DEFINITION 2.1. A face S of a simplicial complex C is *free* if S is included in a unique maximal face of C.

DEFINITION 2.2. Let C be a simplicial complex, and S a free face of C of dimension $<d$. The operation of deleting S and all the faces that include S is called an *elementary d-collapse*. A finite simplicial complex C is called *d-collapsible* if C can be reduced to the void complex by a sequence of elementary d-collapses.

DEFINITION 2.3. A *special elementary d-collapse* is an elementary d-collapse of one of the following types:

(A) Removal of a maximal face of dimension $<d$.

(B) Removal of a free face S of dimension $d-1$ and all the faces that include S.

We shall occasionally say that a special elementary collapse is of type (B)$_r$ ($r\geqq d$), if it is of type (B) and the maximal removed face is r-dimensional.

The following technical lemma was proved in [11].

LEMMA 2.4. [11, lemma 3.2] *Every elementary d-collapse can be effected by a finite sequence of special elementary d-collapses.*

The following fundamental theorem was proved by Wegner [20]:

THEOREM 2.5. *Every d-representable complex is d-collapsible.*

REMARK. The effect of an elementary d-collapse on the h-vector of a simplicial complex C was computed in Eckhoff's paper [6]. This effect is very simple if we are dealing with a special elementary d-collapse.

Indeed if C' is obtained from C by removing a maximal face of dimension e, $e < d$ then $h_e(C) = h_e(C') + 1$, and $h_k(C) = h_k(C')$ for all $k \neq e$.

If C' is obtained from C by a special elementary collapse of type $(\mathrm{B})_r$, $r \geqq d$, then

$$h_k(C) = h_k(C') + 1 \qquad \text{for } d-1 \leqq k \leqq r \quad \text{and}$$

$$h_k(C) = h_k(C') \qquad \text{otherwise.}$$

This fact obviously implies (1.7)(i). Moreover, it implies that if C is reduced to $\varnothing$ by a sequence of special elementary d-collapses, then the number of steps of type $(\mathrm{B})_k$ $(k \geqq d)$ is precisely $h_k(C) - h_{k+1}(C)$. This shows that the sequence $(h_k(C) : k \geqq d-1)$ is non-decreasing. Inequality (1.7)(iii) can be viewed as a strengthening of this last observation.

However, we shall not use this combinatorial interpretation of the numbers h_j. In our proof they will appear as dimensions of some linear spaces.

3. Algebraic background

The material presented here is self-contained and seems sufficient for our purposes. The reader interested in a fuller treatment of exterior algebra may consult [15] or [2].

Let $V = E^n$ be the n-dimensional Euclidean space, with the fixed standard orthonormal basis $(e_1, \ldots, e_n)$. Put $N = \{1, 2, \ldots, n\}$; the exterior algebra ΛV is a 2^n-dimensional inner product space with orthonormal basis $\{e_S : S \subset N\}$. A multilinear associative multiplication $\wedge$ on ΛV with identity $1 = e_\varnothing$ is defined by the following rules:

(a) if $S = \{i_1, \ldots, i_s\} \subset N$, $i_1 < i_2 < \cdots < i_s$, then $e_S = e_1 \wedge \cdots \wedge e_{i_s}$ (here e_i is an abbreviation for $e_{\{i\}}$).

(b) $e_i \wedge e_j = -e_j \wedge e_i$ for all $i, j \in N$.

For $0 \leqq k \leqq n$, $\Lambda^k V$ is the subspace of V spanned by $\{e_S : S \in N^{(k)}\}$.

If $(f_1, \ldots, f_n)$ is another basis of V and $S = \{i_1, \ldots, i_s\} \subset N$, $i_1 < \cdots < i_s$, write $f_S = f_{i_1} \wedge \cdots \wedge f_{i_s}$. It is easy to show that the set $\{f_S : S \in N^{(k)}\}$ is a basis of $\Lambda^k V$.

Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be the transition matrix from the standard basis $(e_1, \ldots, e_n)$ to $(f_1, \ldots, f_n)$, i.e., $f_i = \Sigma\{a_{ij}e_j : j \in N\}$ for $i \in N$. For $S, T \subset N$ put $A_{S|T} = (a_{ij})_{i \in S, j \in T}$. The transition from the basis $\{e_S : S \in N^{(k)}\}$ of $\Lambda^k V$ to $\{f_S : S \in N^{(k)}\}$ is given by

$$f_S = \sum \{(\det A_{S|T}) e_T : T \in N^{(k)}\} \qquad \text{for } S \in N^{(k)}. \tag{3.1}$$

Put $M = \{1, 2, \ldots, m\}$. For an arbitrary $m \times n$ matrix A, and for $1 \leqq k \leqq$

$\min(m,n)$, denote by $\mathbf{C}_k(A)$ the matrix $(\det A_{S|T})_{S\in M^{(k)},T\in N^{(k)}}$. $\mathbf{C}_k(A)$ is called the k-th compound of the matrix A (see [9, p. 19]). From the discussion above it follows that if A is square and non-singular, then $\mathbf{C}_k(A)$ is non-singular as well. Therefore, if the columns of a rectangular matrix A are linearly independent, then so are the columns of $\mathbf{C}_k(A)$.

Let $(f_1,\dots,f_n)$ be an orthonormal basis of V. It follows from the Cauchy–Binet Theorem ([9, p. 9]) that the basis $\{f_S : S\in N^{(k)}\}$ of $\wedge^k V$ is also orthonormal.

For $g, f\in\wedge V$, the *left interior product* $g \llcorner f\in\wedge V$ is defined by the requirement

$$\langle u, g\llcorner f\rangle = \langle u\wedge g, f\rangle \qquad \text{for all } u\in\wedge V. \tag{3.2}$$

Clearly $g\llcorner f$ is a bilinear function of f and g. If $g\in\wedge^d V$ and $f\in\wedge^{k+d}V$ then $g\llcorner f\in\wedge^k V$. It follows from the definition that

$$h\llcorner(g\llcorner f) = (h\wedge g)\llcorner f. \tag{3.3}$$

If $(f_1,\dots,f_n)$ is any orthonormal basis of V, then

$$f_T\llcorner f_S = \begin{cases} \pm f_{S\setminus T} & \text{if } S\supset T,\\ 0 & \text{if } S\not\supset T.\end{cases} \tag{3.4}$$

Note that if $f=\sum_{i=1}^n \alpha_i e_i\in V$ and $S=\{i_1,\dots,i_s\}$ where $i_1<i_2<\cdots<i_s$, then

$$f\llcorner e_S = \sum_{j=1}^{s}(-1)^{s-j}\alpha_{i_j}e_{S\setminus\{i_j\}}. \tag{3.5}$$

Thus $\llcorner$-multiplying on the left by f is a weighted boundary operator, which reduces to a usual boundary operator when $f=\sum_{i=1}^n e_i$.

For subspaces U_1, U_2 of $\wedge V$ define

$$U_1\wedge U_2 = \operatorname{span}\{u_1\wedge u_2 : u_1\in U_1,\ u_2\in U_2\}, \tag{3.6}$$

$$U_1\llcorner U_2 = \operatorname{span}\{u_1\llcorner u_2 : u_1\in U_1,\ u_2\in U_2\}. \tag{3.7}$$

Note that

$$U_1\wedge(U_2+U_3) = U_1\wedge U_2 + U_1\wedge U_3, \tag{3.8}$$

$$U_1\llcorner(U_2+U_3) = U_1\llcorner U_2 + U_1\llcorner U_3, \tag{3.9}$$

and

$$U_1\llcorner(U_2\llcorner U_3) = (U_1\wedge U_2)\llcorner U_3. \tag{3.10}$$

A linear subspace U of $\wedge V$ is a $\wedge$-*ideal* [*left* L-*ideal*] if $g \wedge u \in U$ [resp. $g \llcorner u \in U$] for all $g \in \wedge V$, $u \in U$. A subspace U of $\wedge V$ is *graded* if $U = \sum_{i=0}^{n} (U \cap \wedge^i V)$.

A basis $(f_1, \dots, f_n)$ of V is *in general position* (with respect to the standard basis $(e_1, \dots, e_n)$) if every square submatrix of the transition matrix A from $(e_1, \dots, e_n)$ to $(f_1, \dots, f_n)$ is non-singular. In this case we have, for S, $T \subset N$, $|S| = |T| = r$,

$$\det A_{S|T} = \langle f_S, e_T \rangle \neq 0, \tag{3.11}$$

and therefore, by previous remarks,

$$\det \mathbf{C}_k(A_{S|T}) \neq 0 \qquad \text{for every } k, \quad 1 \leqq k \leqq r. \tag{3.12}$$

Another useful application of (3.11) is the following: Let $F, R, T \subset N$, $|F| + |R| = |T|$. Then

$$\langle e_F, f_R \llcorner e_T \rangle = \langle e_F \wedge f_R, e_T \rangle = \begin{cases} 0 & \text{if } T \not\supset F, \\ \pm \langle f_R, e_{T \setminus F} \rangle \neq 0 & \text{if } T \supset F. \end{cases} \tag{3.13}$$

REMARK. In most textbooks (such as [15]) it is not assumed that $\wedge V$ is an inner product space, and the left (and right) interior products $g \llcorner f$ (resp. $g \lrcorner f$) are defined for $f \in \wedge V$ and $g \in \wedge V^*$, where $(\wedge V)^* = \wedge (V^*)$ is the dual space of $\wedge V$. Use of this definition would not change anything essential, but would slightly complicate the notation.

4. Modules associated with simplicial complexes

Let C be a simplicial complex on the vertex set $[n] (= \{1, 2, \dots, n\})$. Let V be a real inner product space with standard basis $(e_1, \dots, e_n)$, and let $(f_1, \dots, f_n)$ be another basis of V.

DEFINITION 4.1.

(1) $M(C) = \operatorname{span}\{e_S : S \in C\} \subset \wedge V$.
Note that $M(C)$ is a graded left L-ideal of $\wedge V$.

(2) $M_k(C) = M(C) \cap \wedge^{k+1} V (= \operatorname{span}\{e_S : S \in C_k\})$.

(3) $\bar{M}(C) = \operatorname{span}\{e_S : S \notin C\}$.
Note that $\bar{M}(C)$ is a graded $\wedge$-ideal of $\wedge V$.

(4) $\bar{M}_k(C) = \bar{M}(C) \cap \wedge^{k+1} V$.

(5) $I(C) = \wedge V / \bar{M}(C)$.
For $m \in \wedge V$ we write $\tilde{m} = m + \bar{M}(C) \in I(C)$.

(6) $I_k(C)=\{\tilde{m} : m \in \wedge^{k+1}V\} \approx \wedge^{k+1}V/\tilde{M}_k(C)$.

Obviously $\dim M_k(C)=\dim I_k(C)=f_k(C)$.

Note that $\{\tilde{e}_S : S \in C\}$ is a basis of $I(C)$. It follows that $I(C)$ is the direct sum of $I_k(C)$, $-1 \leqq k \leqq n-1$.

Next we define an order relation $<$ (the opposite lexicographic order) on the subsets of $[n]$, as follows:

Definition 4.2. Suppose $S=\{i_1,\dots,i_s\}$, $T=\{j_1,\dots,j_t\}$, where $i_1<i_2<\cdots<i_s$ and $j_1<j_2<\cdots<j_t$. $S<T$ if $s<t$, or if $s=t$ and there exists an m such that $i_k=j_k$ for all $k<m$ and $i_m<j_m$. (I.e., $S<T$ iff $|S|<|T|$ or $|S|=|T|$ and $\min(S \triangle T) \in S$.)

For basis elements f_S, $f_T (S, T \subset [n])$ $f_S < f_T$ iff $S<T$.

Note that $I(C)$ is spanned by the set $\{\tilde{f}_S : S \subset N\}$. Define

$$D=\{S \subset N,\ \tilde{f}_S \notin \operatorname{span}\{\tilde{f}_T : T \subset [n],\ T<S\}\}.$$

One can easily verify that the set $\{\tilde{f}_S : S \in D\}$ is a basis of $I(C)$.

In fact, the following assertion holds:

Proposition 4.3. *If E is an initial segment of $[n]^{(k)}(=\{S \subset [n] : |S|=k\})$ with respect to the order relation $<$, then the set $\{\tilde{f}_S : S \in D \cap E\}$ is a linear basis of* $\operatorname{span}(\tilde{f}_S : S \in E\}$.

The verification of Proposition 4.3 is straightforward and is omitted.

Proposition 4.4. (compare [18]) *D is a simplicial complex.*

Proof. First note that if R, S, $T \subset [n]$, $S<T$ and $R \cap (S \cup T)=\emptyset$, then $R \cup S < R \cup T$. Now suppose $S_1 \in D$, $S_2 \subset S_1$ and $S_2 \notin D$.

Since $S_2 \notin D$, we can write

$$\tilde{f}_{S_2}=\sum\{\alpha_S \tilde{f}_S : S<S_2\}. \tag{4.1}$$

Put $R=S_1 \setminus S_2$ and multiply both sides of (4.1) by $\tilde{f}_R$. Since $\tilde{f}_R \wedge \tilde{f}_S=0$ unless $R \cap S=\emptyset$, we obtain

$$\tilde{f}_{S_1}=\tilde{f}_{R \cup S_2}=\pm \tilde{f}_R \wedge \tilde{f}_{S_2}=\pm \sum\{\pm \alpha_S \tilde{f}_{R \cup S} : S<S_2,\ S \cap R=\emptyset\}.$$

Thus $\tilde{f}_{S_1} \in \operatorname{span}\{\tilde{f}_T : T \subset [n],\ T<S_1\}$, in contradiction to $S_1 \in D$. ∎

Obviously $f_k(C)=f_k(D)$ for all $k \geqq 0$.

REMARKS.

(1) If $\bar{M}$ is any graded $\wedge$-ideal of $\wedge V$, define $I(\bar{M}) = \wedge V/\bar{M}$. One can define D in exactly the same way as done here for $I(C)$, and Propositions 4.3 and 4.4 remain valid.

(2) The complex D depends on the basis $(f_1, \ldots, f_n)$ (but "generically" it is independent of the basis).

5. Generalized homology groups

In this section we define generalized homology groups $H_k^r(C)$ for a simplicial complex C. The main result in this section is that some of these groups vanish for d-collapsible complexes. This is a crucial step in proving that the f-vectors of d-collapsible complexes satisfy Eckhoff's inequalities.

As before, let V be a real inner product space with a standard basis $(e_1, \ldots, e_n)$, and let $(f_1, \ldots, f_n)$ be another basis of V. For $0 \leqq r \leqq n$ define $F_r = \operatorname{span}\{f_1, \ldots, f_r\} \subset V$. Recall that $[r] = \{1, 2, \ldots, r\}$.

Let M be a graded left $\llcorner$-ideal of $\wedge V$. For $-1 \leqq i \leqq n-1$ define $M_i = M \cap \wedge^{i+1} V$.

DEFINITION 5.1.

$$Z_k^r = \{x \in M_k : f_{[r]} \llcorner x = 0\}, \tag{5.1}$$

$$B_k^r = F_r \llcorner M_{k+1} \quad (\subset M_k). \tag{5.2}$$

PROPOSITION 5.2. $B_k^r \subset Z_k^r$.

PROOF. For $1 \leqq i \leqq r$, $m \in M_{k+1}$,

$$f_{[r]} \llcorner (f_i \llcorner m) = (f_{[r]} \wedge f_i) \llcorner m = 0 \llcorner m = 0.$$

DEFINITION 5.3.

$$H_k^r(M) = Z_k^r / B_k^r. \tag{5.3}$$

If C is a simplicial complex with vertex set $[n]$, put $H_k^r(C) = H_k^r(M(C))$.

From here throughout the rest of the paper we assume that $(f_1, \ldots, f_n)$ is a *fixed orthonormal basis* of V, in general position with respect to $(e_1, \ldots, e_n)$. The existence of such a basis is easily proved, see [11, sec. 2]. All the applications of the definitions in Sections 4 and 5 will be taken with respect to this fixed basis.

REMARKS.

(1) If $r = 1$ and $f_1 = e_1 + \cdots + e_n$, then $H_k^1(C)$ coincides with the usual homology group $H_k(C, \mathbf{R})$.

(2) For our purposes condition (3.11) on the basis $(f_1, \dots, f_n)$ can be relaxed. It can be replaced by

$$\text{(5.4)} \qquad \det A_{[r][T} = \langle f_{[r]}, e_T \rangle \neq 0 \qquad \text{for all } 0 < r \leqq n \text{ and all } T \in N^{(r)}.$$

As remarked at the end of Section 3, the orthogonality of the basis $(f_1, f_2, \dots, f_n)$ is, in a sense, just a matter of convenience.

(3) It may be true that the generalized homology groups $H_k^r(C)$ defined above are independent of the particular choice of the basis $(f_1, \dots, f_n)$ (provided it satisfies (5.4)). We know that this is true in the "generic" case. However, failure of $(f_1, \dots, f_n)$ to satisfy (5.4) may affect the groups $H_k^r(C)$. Note that for fixed r, $H_k^r(M)$ actually depends on $F_r = \operatorname{span}\{f_1, \dots, f_r\}$, and not on the individual vectors $f_1, \dots, f_r$.

The main result of this section is the following theorem.

THEOREM 5.4. *Suppose C is a d-collapsible complex; then $H_{d+k}^{k+1}(C) = 0$ for all $k \geqq 0$.*

The proof of Theorem 5.4 is based upon the following proposition.

Let d, b, c be fixed integers, $d \geqq 0$, $0 \leqq b \leqq c \leqq n - d$. Put $i = c - b$. Let F, G be two subsets of $[n]$, $F \subset G$, $|F| = d$, $|G| = d + c$. For each set S such that $F \subset S \subset G$ and $|S| = d + b$, let α_S be a real number.

PROPOSITION 5.5. *There exists a unique $l \in \wedge^i F_c$ such that*

$$\text{(5.5)} \qquad \langle e_S, l \llcorner e_G \rangle = \alpha_S \qquad \textit{for all } S \textit{ such that } F \subset S \subset G, \quad |S| = d + b.$$

PROOF. The vectors $\{f_T : T \in [c]^{(i)}\}$ form a basis of $\wedge^i F_c$. Writing $l = \Sigma\{\xi_T f_T : T \in [c]^{(i)}\}$ condition (5.5) reduces to the following system of linear equations:

$$\sum \{\langle e_S, f_T \llcorner e_G \rangle \xi_T : T \in [c]^{(i)}\} = \alpha_S \qquad (F \subset S \subset G, \ |S| = d + b).$$

Thus the proof of our proposition is reduced to showing that the coefficient matrix D,

$$\text{(5.6)} \qquad D_{T,S} = \langle e_S, f_T \llcorner e_G \rangle \quad \text{for } F \subset S \subset G, \ |S| = d + b \ \text{ and } \ T \in [c]^{(i)},$$

is square and non-singular.

Clearly D is a $\binom{c}{i} \times \binom{c}{i}$ matrix. Now,

$$D_{T,S} = \langle e_S, f_T \llcorner e_G \rangle = \langle e_S \wedge f_T, e_G \rangle = \langle e_S \wedge e_{G \setminus S}, e_G \rangle \langle f_T, e_{G \setminus S} \rangle$$

(see (3.13)). Write $\varepsilon(S) = \langle e_S \wedge e_{G \setminus S}, e_G \rangle$, and note that $\varepsilon(S) = \pm 1$. Therefore, by

(3.1), $D_{T,S} = \varepsilon(S)\det A_{T|G\setminus S}$. Thus D is obtained from the i-th compound matrix $\mathbf{C}_i(A_{[c]|G\setminus F})$ by renaming the columns $(G \setminus S \to S)$ and multiplying some of them by -1. Since $A_{[c]|G\setminus F}$ is non-singular (by our assumption (3.11)), it follows that $\mathbf{C}_i(A_{[c]|G\setminus F})$ is non-singular as well, and so is D. ∎

Proof of Theorem 5.4. Assume that C is d-collapsible. We must show that $H^{k+1}_{d+k}(C)=0$ $(k \geqq 0)$, or, in other words: If $m \in M_{d+k}(C)$ and $f_{[k+1]} \llcorner m = 0$ then $m \in F_{k+1} \llcorner M_{d+k+1}(C)$.

Suppose $C \searrow C' \searrow C'' \searrow \cdots \searrow C^{(t)} = \varnothing$ is a sequence of special elementary d-collapses that reduces C to the void complex. The proof is by induction on t. We assume the validity of the theorem for C' and prove it for C.

Let F be the free face and let G be the unique maximal face that includes F in the special elementary d-collapse $C \searrow C'$. If $|F| \leqq d$ and $G = F$ there is nothing to prove. Assume $|F| = d$ and $\dim G = d + k + i$. If $i < 0$ then $M_{d+j}(C) = M_{d+j}(C')$ for all $j \geqq k$, and there is again nothing to prove. Suppose $i \geqq 0$. Assume $m \in M_{d+k}(C)$, $f_{[k+1]} \llcorner m = 0$. If $\langle m, e_S\rangle = 0$ for all $S \in C$ s.t. $F \subset S$, then $m \in M_{d+k}(C')$, and by the induction hypothesis $m \in F_{k+1} \llcorner M_{d+k+1}(C') \subset F_{k+1} \llcorner M_{d+k+1}(C)$. Hence we may assume that $\langle m, e_S\rangle \neq 0$ for some $S \in C$ s.t. $F \subset S$.

Case I. $i = 0$, i.e., $\dim G = d + k$. In this case $\langle m, e_G\rangle = \alpha_G \neq 0$. We will prove that in this case $f_{[k+1]} \llcorner m \neq 0$, by showing that $\langle e_F, f_{[k+1]} \llcorner m\rangle \neq 0$. Suppose

$$m = \sum\{\alpha_S e_S : S \in C,\ \dim S = d + k\}.$$

If $S \neq G$, then $F \not\subset S$, and therefore $\langle e_F, f_{[k+1]} \llcorner e_S\rangle = 0$ (see (3.13)). Thus

$$\begin{aligned}\langle e_F, f_{[k+1]} \llcorner m\rangle &= \alpha_G \langle e_F, f_{[k+1]} \llcorner e_G\rangle \\ &= \alpha_G \langle e_F \wedge f_{[k+1]}, e_G\rangle = \pm\alpha_G \langle f_{[k+1]}, e_{G\setminus F}\rangle \neq 0,\end{aligned}$$

a contradiction (see (3.13)).

Case II. $i > 0$. The proof in this case proceeds along the following lines:

(a) Split m into a sum $m = m_1 + m_2$, where $m_1 \in M_{d+k}(C)$ is of the form $l \llcorner e_G$, and $m_2 \in M_{d+k}(C')$.

(b) Show that $m_1 \in F_{k+1} \llcorner M_{d+k+1}(C)$.

The rest is easy. Since by (b) $m_1 \in F_{k+1} \llcorner M_{d+k+1}(C)$, it follows from Proposition 5.2 that $f_{[k+1]} \llcorner m_1 = 0$. Since $f_{[k+1]} \llcorner m = 0$, also $f_{[k+1]} \llcorner m_2 = 0$. Since $m_2 \in M_{d+k}(C')$, the induction hypothesis implies that $m_2 \in F_{k+1} \llcorner M_{d+k+1}(C') \subset F_{k+1} \llcorner M_{d+k+1}(C)$. By (b) also $m = m_1 + m_2 \in F_{k+1} \llcorner M_{d+k+1}(C)$.

We now perform steps (a) and (b) as described above.

Step (a). As before, assume that $m = \Sigma\{\alpha_S e_S : S \in C, \dim S = d + k\}$. Proposition 5.5 (with the substitutions $b \to k+1$, $c \to k+i+1$) asserts the existence of an $l \in \wedge^i F_{k+i+1}$ such that

$$\langle e_S, l \llcorner e_G\rangle = \alpha_S \qquad \text{for all } S \text{ such that } F \subset S \subset G, \quad \dim S = d + k.$$

Now put $m_1 = l \llcorner e_G$, $m_2 = m - m_1$. m_1 is of the form $\Sigma\{\beta_S e_S : S \subset G, \dim S = d + k\}$, and $\beta_S = \alpha_S$ if $F \subset S \subset G$. Therefore $m_1 \in M_{d+k}(C)$ and $m_2 \in M_{d+k}(C')$.

Step (b). Write

$$l = \sum\{\lambda_R f_R : R \in [k+i+1]^{(i)}\}, \tag{5.7}$$

and put $\lambda^* = \lambda_{[k+2,k+i+1]}$ ($[k+2, k+i+1] = \{k+2, \ldots, k+i+1\}$). Note that if $R \in [k+i+1]^{(i)}$ and $R \neq [k+2, k+i+1]$ then $R \cap [k+1] \neq \varnothing$ and therefore $f_{[k+1]} \wedge f_R = 0$. It follows that $f_{[k+1]} \wedge l = \lambda^* \cdot f_{[k+1]} \wedge f_{[k+2,k+i+1]} = \lambda^* \cdot f_{[k+i+1]}$. Now, since by our assumption $f_{[k+1]} \llcorner m = 0$, we have

$$\begin{aligned} 0 = \langle e_F, f_{[k+1]} \llcorner m\rangle &= \langle e_F, f_{[k+1]} \llcorner \sum\{\alpha_S e_S : S \in C, \dim S = d+k\}\rangle \\ &= \langle e_F, f_{[k+1]} \llcorner \sum\{\alpha_S e_S : F \subset S \subset G, \dim S = d+k\}\rangle. \end{aligned} \tag{5.8}$$

(If $F \not\subset S$, then $\langle e_F, f_{[k+1]} \llcorner e_S\rangle = 0$, and if $F \subset S \in C$ then $S \subset G$.) By the same token

$$\langle e_F, f_{[k+1]} \llcorner m_1\rangle = \langle e_F, f_{[k+1]} \llcorner \sum\{\beta_S e_S : F \subset S \subset G, \dim S = d+k\}\rangle.$$

Since $\alpha_S = \beta_S$ for all $F \subset S \subset G$, we obtain:

$$\begin{aligned} 0 = \langle e_F, f_{[k+1]} \llcorner m\rangle &= \langle e_F, f_{[k+1]} \llcorner m_1\rangle \\ &= \langle e_F, f_{[k+1]} \llcorner (l \llcorner e_G)\rangle = \langle e_F, (f_{[k+1]} \wedge l) \llcorner e_G\rangle \\ &= \lambda^* \cdot \langle e_F, f_{[k+i+1]} \llcorner e_G\rangle = \lambda^* \cdot \langle e_F \wedge f_{[k+i+1]}, e_G\rangle \\ &= \pm\lambda^* \cdot \langle f_{[k+i+1]}, e_{G\setminus F}\rangle. \end{aligned}$$

Since $\langle f_{[k+i+1]}, e_{G\setminus F}\rangle \neq 0$ by our assumption (3.11)), we conclude that $\lambda^* = 0$. It follows by (5.7) that

$$l \in \operatorname{span}\{f_R : R \in [k+i+1]^{(i)}, R \neq [k+2, k+i+1]\} = F_{k+1} \wedge \wedge^{i-1} F_{k+i+1}.$$

Now

$$\begin{aligned} m_1 = l \llcorner e_G &\in (F_{k+1} \wedge \wedge^{i-1} F_{k+i+1}) \llcorner M_{d+k+i}(C) \\ &= F_{k+1} \llcorner (\wedge^{i-1} F_{k+i+1} \llcorner M_{d+k+i}(C)) \subset F_{k+1} \llcorner M_{d+k+1}(C). \end{aligned}$$

This completes step (b), and with it the proof of Theorem 5.4. ∎

Theorem 5.4 can be actually strengthened, as follows. Let C' be a subcomplex of C. The inclusion map $i = i_k : M_k(C') \to M_k(C)$ map $Z_k^l(C')$ into $Z_k^l(C)$ and $B_k^l(C')$ into $B_k^l(C)$, and therefore induces a homomorphism $\varphi(= \varphi_k^l) : H_k^l(C') \to H_k^l(C)$.

THEOREM 5.6. *If C' is obtained from C by a special elementary d-collapse, then $\varphi : H_{d+k}^{k+1}(C') \to H_{d+k}^{k+1}(C)$ is an isomorphism for every $k \geqq 0$.*

(Theorem 5.6 is not needed for the proof of Eckhoff's conjecture.)

PROOF. For $m \in Z_k^l(C)$, put $\tilde{m} = m + B_k^l(C) \in H_k^l(C)$. Let $k \geqq 0$ be fixed. Let F be the free face and let G be the unique maximal face that includes F in the special elementary d-collapse $C \searrow C'$. We may assume that $\dim G \geqq d + k$. Exzctly as in the proof of Theorem 5.4 one can show that if $m \in Z_{d+k}^{k+1}(C)$, then $m = m_1 + m_2$, where $m_1 \in B_{d+k}^{k+1}(C)$ and $m_2 \in Z_{d+k}^{k+1}(C')$. Thus $\tilde{m} = \tilde{m}_2$, and $\tilde{m}_2$ is clearly in the range of φ_{d+k}^{k+1}. This shows that φ_{d+k}^{k+1} is an epimorphism.

It remains to show that φ_{d+k}^{k+1} is a monomorphism, or equivalently, that $B_{d+k}^{k+1}(C) \cap Z_{d+k}^{k+1}(C') \subset B_{d+k}^{k+1}(C')$. (The opposite inclusion is always true.) This relation is trivial when $\dim G = d + k$, so we may assume that $\dim G > d + k$. Suppose $m \in B_{d+k}^{k+1}(C) \cap Z_{d+k}^{k+1}(C')$. Since $m \in B_{d+k}^{k+1}(C) (= F_{k+1} \llcorner M_{d+k+1}(C))$, m can be written as a sum:

$$m = \Sigma \{a_T \llcorner e_T : T \in C, \dim T = d + k + 1\}, \tag{5.9}$$

where $a_T \in F_{k+1}$ for every T. Define

$$m_1 = \sum \{a_T \llcorner e_T : F \subset T \subset G, \dim T = d + k + 1\}. \tag{5.10}$$

Clearly $m - m_1 = \Sigma \{a_T \llcorner e_T : T \in C', \dim T = d + k + 1\} \in B_{d+k}^{k+1}(C')$. Thus $m - m_1 \in Z_{d+k}^{k+1}(C')$, and since by our assumption $m \in Z_{d+k}^{k+1}(C')$, it follows that $m_1 \in Z_{d+k}^{k+1}(C')$, and we only have to prove that $m_1 \in B_{d+k}^{k+1}(C')$. By Proposition 5.5 (with the substitution $b \to k+2$, $c \to k+i+1$), for every T such that $F \subset T \subset G$ and $\dim T = d + k + 1$ there exists a unique element $l_T \in \Lambda^{i-1} F_{k+i+1}$ such that $\langle e_R, l_T \llcorner e_G \rangle = \delta_{T,R}$ for all R such that $F \subset R \subset G$ and $\dim R = d + k + 1$. Thus $l_T \llcorner e_G = e_T + c_T$, where $c_T \in M_{d+k+1}(C')$. Define

$$l = \sum \{a_T \wedge l_T : F \subset T \subset G, \dim T = d + k + 1\}, \tag{5.11}$$

and

$$m_2 = \sum \{a_T \llcorner c_T : F \subset T \subset G, \dim T = d + k + 1\}. \tag{5.12}$$

Note that $l \in F_{k+1} \wedge \Lambda^{i-1} F_{k+i+1} \subset \Lambda^i F_{k+i+1}$ and

$$m_2 \in F_{k+1} \llcorner M_{d+k+1}(C') = B_{d+k}^{k+1}(C') \subset Z_{d+k}^{k+1}(C').$$

Note also that, for all T such that $F \subset T \subset G$ and $\dim T = d + k + 1$,

$$(a_T \wedge l_T) \llcorner e_G = a_T \llcorner (l_T \llcorner e_G) = a_T \llcorner (e_T + c_T) = a_T \llcorner e_T + a_T \llcorner c_T.$$

It follows from (5.11), (5.10) and (5.12) that $l \llcorner e_G = m_1 + m_2$. Since m_1 and m_2 belong to $Z^{k+1}_{d+k}(C')$, it follows that $l \llcorner e_G \in Z^{k+1}_{d+k}(C')$, and therefore $\langle l \llcorner e_G, e_S \rangle = 0$ for every S such that $F \subset S \subset G$ and $\dim S = d + k$. From the uniqueness part of Proposition 5.5 (with $b \to k + 1$, $c \to k + i + 1$) it follows that $l = 0$ and therefore $l \llcorner e_G = m_1 + m_2 = 0$. Therefore $m_1 = -m_2 \in B^{k+1}_{d+k}(C')$. ∎

REMARK. Call two complexes C, D *d-collapse equivalent* if D can be obtained from C by a finite sequence of elementary d-collapses and "inverse elementary d-collapses". (Compare [3, ch. 1].) Theorem 5.6 actually shows that if C and D are d-collapse equivalent then $H^{k+1}_{d+k}(C) \approx H^{k+1}_{d+k}(D)$ for every $k \geqq 0$.

It can be shown that if C and D are d-collapse equivalent then there is a complex E such that C and D d-collapse to E.

6. Proof of the Main Theorem

In view of Theorem 5.4, the "only if" part of Eckhoff's conjecture follows from the following theorem:

THEOREM 6.1. *Let $d \geqq 1$ be a fixed integer. Suppose C is a finite simplicial complex satisfying*

$$H^{k+1}_{d+k}(C) = 0 \qquad \textit{for all } k \geqq 0. \tag{6.1}$$

Let $h(C) = (h_0, h_1, \ldots)$ be the h-vector of C (see (1.5)). Then

(i) $h_k \geqq 0, \qquad k = 0, 1, \ldots,$
(ii) $h_k^{(k+1)} \leqq h_{k-1}, \qquad k = 1, 2, \ldots, d - 1,$
(iii) $h_k^{(d)} \leqq h_{k-1} - h_k, \qquad k = d, d + 1, \ldots.$

REMARK. (ii) is just the Kruskal–Katona Theorem and (i) is obvious for $k < d$. Thus we have to prove only inequality (i) for $k \geqq d$ and inequality (iii).

Since the proof is somewhat involved we first describe its main steps.

First we choose for every $k \geqq 0$ a subspace of $M_{d+k} (= M_{d+k}(C))$ which is a complement of Z^{k+1}_{d+k} and denote it by M^0_{d+k}.

Using the fact that $H^{k+1}_{d+k}(C) = 0$ we show that

$$Z^{k+1}_{d+k} = B^{k+1}_{d+k} = F_{k-1} \llcorner M_{d+k+1} = \sum_{j>0} \bigwedge^{j} F_{k+j} \llcorner M^0_{d+k+j}.$$

Moreover, we prove that

$$\dim Z^{k+1}_{d+k} = \sum_{j>0} \dim\left(\bigwedge^{j} F_{k+j} \llcorner M^0_{d+k+j}\right) = \sum_{j>0} \binom{k+j}{j} \dim M^0_{d+k+j}.$$

From this equality we conclude that $\dim M^0_{d+k} = h_{d+k}$,[†] which proves inequality (i) for $k \geqq d$.

To prove inequality (ii) we consider the simplicial complex D defined in Section 4. We prove that

$$|\{S \in D : |S| = d+k+1,\ S \supset [k+1]\}| = \dim M^0_{d+k} = h_{d+k}.$$

From the Kruskal–Katona Theorem it follows that

$$h^{(d)}_{d+k} \leqq |\{S \in D : |S| = d+k,\ S \supset [k+1]\}|.$$

The proof is finished by a counting argument which shows that

$$|\{S \in D : |S| = d+k,\ S \supset [k+1]\}| \leqq h_{d+k-1} - h_{d+k}.$$

Proof of Theorem 6.1. Suppose $\dim C = d + r$. Recall that Z^{k+1}_{d+k} is a subspace of M_{d+k} defined by

$$Z^{k+1}_{d+k} = \{x \in M_{d+k} : f_{[k+1]} \llcorner x = 0\}.$$

For $k \geqq 0$ choose M^0_{d+k} to be a subspace of M_{d+k} complementary to Z^{k+1}_{d+k}. ($M^0_{d+k} + Z^{k+1}_{d+k} = M_{d+k}$, $M^0_{d+k} \cap Z^{k+1}_{d+k} = \{0\}$.) The following assertion follows immediately from the definition of M^0_{d+k}.

Assertion (1). If $x \in M^0_{d+k}$, $x \neq 0$, then $f_{[k+1]} \llcorner x \neq 0$.

Next we prove a few more assertions:

Assertion (2).

$$Z^{k+1}_{d+k} = \sum_{j>0} \bigwedge^{j} F_{k+j} \llcorner M^0_{d+k+j}. \tag{6.2}$$

Proof. By decreasing induction on k. By our assumption $H^{k+1}_{d+k}(C) = 0$ for all $k \geqq 0$. This means that $Z^{k+1}_{d+k} = B^{k+1}_{d+k} = F_{k+1} \llcorner M_{d+k+1}$. In particular, since $\dim C = d + r$ we obtain $Z^{r+1}_{d+r} = B^{r+1}_{d+r} = 0$. Thus (6.2) holds for $k = r$. Now suppose (6.2) holds for $k + 1$ and prove it for k, as follows:

[†] For the case of a d-collapsible complex C, the equality $\operatorname{codim} Z^{k+1}_{d+k} = h_{d+k}$ follows easily from the proof of Theorem 5.4.

$$\begin{aligned}
Z^{k+1}_{d+k} = B^{k+1}_{d+k} &= F_{k+1} \llcorner M_{d+k+1} = F_{k+1} \llcorner (M^0_{d+k+1} + Z^{k+2}_{d+k+1}) \\
&= F_{k+1} \llcorner M^0_{d+k+1} + F_{k+1} \llcorner Z^{k+2}_{d+k+1} \\
&= F_{k+1} \llcorner M^0_{d+k+1} + F_{k+1} \llcorner \left(\sum_{j \geq 0} \bigwedge^{j} F_{k+1+j} \wedge M^0_{d+k+1+j} \right) \\
&= F_{k+1} \llcorner M^0_{d+k+1} + \sum_{j>0} F_{k+1} \llcorner \left(\bigwedge^{j} F_{k+1+j} \llcorner M^0_{d+k+1+j} \right) \\
&= F_{k+1} \llcorner M^0_{d+k+1} + \sum_{j>0} \left(F_{k+1} \wedge \bigwedge^{j} F_{k+1+j} \right) \llcorner M^0_{d+k+1+j} \\
&= F_{k+1} \llcorner M^0_{d+k+1} + \sum_{j>0} \bigwedge^{j+1} F_{k+1+j} \llcorner M^0_{d+k+1+j} \\
&= \sum_{j>0} \bigwedge^{j} F_{k+j} \llcorner M^0_{d+k+j}. \qquad \blacksquare
\end{aligned}$$

For $0 \leqq j \leqq r-k$ and $S \in [k+j]^{(j)}$ (i.e., $S \subset [k+j]$, $|S| = j$), define

$$M_{d+k}[S] = f_S \llcorner M^0_{d+k+j} \quad (\subset M_{d+k}). \tag{6.3}$$

Since $\wedge^j F_{k+j} = \operatorname{span}\{f_S : S \in [k+j]^{(j)}\}$, we have

$$\bigwedge^{j} F_{k+j} \llcorner M^0_{d+k+j} = \sum \{M_{d+k}[S],\ S \in [k+j]^{(j)}\}. \tag{6.4}$$

Define

$$\begin{aligned}
\mathcal{S} = \mathcal{S}(k,r) &= \bigcup \{[k+j]^{(j)} : 0 < j \leqq r-k\} \\
&= \{S : 0 < |S| \leqq r-k,\ S \subset [k+|S|]\}.
\end{aligned} \tag{6.5}$$

ASSERTION (3).

$$Z^{k+1}_{d+k} = \bigoplus \{M_{d+k}[S] : S \in \mathcal{S}\}. \tag{6.6}$$

PROOF. From (6.2) and (6.4) it follows that (6.6) holds with $\oplus$ replaced by Σ. It only remains to show that the sum is direct.

Suppose

$$m = \sum \{m_S : S \in \mathcal{S}\}, \tag{6.7}$$

where $m_S \in M_{d+k}[S]$, and not all the terms m_S are zero. By (6.3) we can write

$$m_S = f_S \llcorner m^0_S, \qquad \text{where } m^0_S \in M^0_{d+k+|S|}. \tag{6.8}$$

We must show that $m \neq 0$. To do this we show that $f_R \llcorner m \neq 0$ for a suitably chosen element $f_R \in \wedge^{k+1} V$.

Choose a set $T \in \mathscr{S}$ of minimum cardinality such that $m_T \neq 0$, and put $t = |T|$. Clearly $T \subset [k+t]$. Define $R = [k+t+1] \setminus T (= [k+t] \setminus T \cup \{k+t+1\})$. Note that if $S \in [k+t]^{(t)}$ then $S \cap R \neq \varnothing$ unless $S = T$. If $S \in [k+j]^{(j)}$ for some $j > t$, then clearly $S \cap R \neq \varnothing$, since $|S| + |R| = j + k + 1$ and $R \subset [k+t+1] \subset [k+j]$. Thus $f_R \wedge f_S = 0$ for all $S = \mathscr{S}$ such that $|S| \geqq t$, except possibly for $S = T$. By the choice of T, $m_S = 0$ for all $S \in \mathscr{S}$ such that $|S| < t$. We conclude that $(f_R \wedge f_S) \llcorner m_S^0 = 0$ for all $S \in \mathscr{S}$, except (perhaps) for $S = T$.

It follows from (6.7) and (6.8) that

$$
\begin{aligned}
f_R \llcorner m &= f_R \llcorner \sum \{m_S : S \in \mathscr{S}\} \\
&= f_R \llcorner \sum \{f_S \llcorner m_S^0 : S \in \mathscr{S}\} \\
&= \sum \{f_R \llcorner (f_S \llcorner m_S^0 : S \in \mathscr{S}\} \\
&= \sum \{(f_R \wedge f_S) \llcorner m_S^0 : S \in \mathscr{S}\} \\
&= (f_R \wedge f_T) \llcorner m_T^0 = \pm f_{R \cup T} \llcorner m_T^0 \\
&= \pm f_{[k+t+1]} \llcorner m_T^0 .
\end{aligned}
$$

Since $0 \neq m_T^0 \in M_{d+k+t}^0$, $f_{[k+t+1]} \llcorner m_T^0 \neq 0$ (Assertion (1)) and thus $m \neq 0$. ■

From Assertion (1) it follows immediately that for every $S \in [k+j]^{(j)}$

(6.9) $$\dim M_{d+k}[S] = \dim f_S \llcorner M_{d+k+j}^0 = \dim M_{d+k+j}^0 .$$

From (6.5), (6.6) and (6.9) we get

(6.10) $$\dim Z_{d+k}^{k+1} = \sum_{j=1}^{r-k} \binom{k+j}{j} \dim M_{d+k+j}^0 .$$

ASSERTION (4).

(6.11) $$\dim M_{d+k}^0 = h_{d+k} .$$

PROOF. By decreasing induction on k.

For $k = r$ ($\dim C = d + r$) we have $Z_{d+r}^{r+1} = H_{d+r}^{r+1}(C) = 0$ and therefore $M_{d+r}^0 = M_{d+r}$ and $\dim M_{d+r}^0 = f_{d+r} = h_{d+r}$.

Suppose (6.11) holds for all $k' > k$. Then, by (6.10) and (1.6)

$$\dim M_{d+k}^0 = \dim M_{d+k} - \dim Z_{d+k}^{k+1} = f_{d+k} - \sum_{j=1}^{r-k} \binom{k+j}{j} h_{d+k+j} = h_{d+k} . \qquad ■$$

Note that from (6.11) it follows that

(6.12) $$\dim Z_{d+k}^{k+1} = f_{d+k} - h_{d+k} .$$

COROLLARY. $h_k \geqq 0$ *for all* $k \geqq d$.

This is part (i) of Theorem 6.1.

To proceed we need the definitions and results of Section 4. There we defined a basis $\{\tilde{f}_S : S \in D\}$ of $I(C)$. This basis is "lexicographically minimal" with respect to the order $<$ on $\{\tilde{f}_S : S \subset N\}$. There we also showed (Proposition 4.3) that D is a simplicial complex and that $f_k(C) = f_k(D)$ for all $k \geqq 0$.

All that is left to prove is inequality (iii), $h_{d+k}^{\langle d \rangle} \leqq h_{d+k-1} - h_{d+k}$ for $0 \leqq k \leqq r$. This will be done by applying the Kruskal–Katona Theorem to the quotient complex $D/[k+1]$ $(=\{S \setminus [k+1], S \in D, [k+1] \subset S\})$. To do this we need a few more assertions.

Define

$$\begin{aligned} E_k(=E(n,d,k)) &= \{S \in [n]^{(d+k+1)} : S \supset [k+1]\} \\ &= \{[k+1] \cup T : T \in [k+2,n]^{(d)}\}. \end{aligned} \tag{6.13}$$

ASSERTION (5).

$$|E_k \cap D| = h_{d+k}. \tag{6.14}$$

PROOF. Define $W = \mathrm{Span}\{f_S : S \in E_k\}$ $(\subset \Lambda^{d+k+1} V)$, $\tilde{W} = \{\tilde{w} : w \in W\} = \mathrm{span}\{\tilde{f}_S : S \in E_k\}$ $(\subset I_{d+k}(C)$, see Definition 4.1). Note that E_k is an initial segment of $[n]^{(d+k+1)}$ with respect to the order relation $<$ (see Definition 4.2), and therefore, by Proposition 4.3, $\{\tilde{f}_S : S \in E_k \cap D\}$ is a linear basis of $\tilde{W}$. Thus it remains to show that

$$\dim \tilde{W} = h_{d+k}. \tag{6.15}$$

But

$$\tilde{W} = W/(W \cap \bar{M}(C)) = W/(W \cap \bar{M}_{d+k}) \approx (W + \bar{M}_{d+k})/\bar{M}_{d+k},$$

and therefore

$$\dim \tilde{W} = \dim(W + \bar{M}_{d+k}) - \dim \bar{M}_{d+k} = \dim(W + \bar{M}_{d+k}) - \tbinom{n}{d+k+1} + f_{d+k}.$$

Therefore (6.15) is equivalent to

$$\dim(W + \bar{M}_{d+k}) = \tbinom{n}{d+k+1} - f_{d+k} + h_{d+k}. \tag{6.16}$$

Define

$$W^{+} = \mathrm{span}\left\{f_S : S \in [n]^{(d+k+1)},\ [k+1] \not\subset S\right\}. \tag{6.17}$$

Since W^+ and M_{d+k} are the orthogonal complements in $\wedge^{d+k+1}V$ of W, resp. $\bar{M}_{d+k}$, (6.16) is equivalent to

$$\dim(W^+ \cap M_{d+k}) = f_{d+k} - h_{d+k}. \tag{6.18}$$

But $m \in W^+ \cap M_{d+k}$ iff $f_{[k+1]} \llcorner m = 0$ (see (3.4)), i.e. $m \in Z_{d+k}^{k+1}(C)$. Thus $W^+ \cap M_{d+k} = Z_{d+k}^{k+1}$, and since $\dim Z_{d+k}^{k+1} = f_{d+k} - h_{d+k}$ ((6.12)), (6.18) follows. ■

For $r \geqq 0$ we use the notation $D_r = \{S \in D : \dim S = r\}$. For $k \geqq -1$ denote $D_{d+k}^0 = D \cap E_k\,(=\{S \in D_{d+k} : [k+1] \subset S\})$.

Proof of Theorem 6.1 (end). Define subsets $\mathcal{S}_0$ and $\mathcal{S}_1$ of D_{d+k-1}^0 as follows:

$$\mathcal{S}_0 = \{S \in D_{d+k-1} : [k+1] \subset S\}, \tag{6.19}$$

$$\mathcal{S}_1 = \{S \setminus \{k+1\} : S \in D_{d+k}^0\}. \tag{6.20}$$

Clearly $\mathcal{S}_0$ and $\mathcal{S}_1$ are disjoint. From Assertion (5) it follows that $|\mathcal{S}_1| = |D_{d+k}^0| = h_{d+k}$, therefore

$$|\mathcal{S}_0| \leqq |D_{d+k-1}^0| - |\mathcal{S}_1| = h_{d+k-1} - h_{d+k}. \tag{6.21}$$

Note that $|\mathcal{S}_0|$ is the number of $(d-2)$-faces in the quotient complex $D^* = D/[k+1]$. The number of $(d-1)$-faces of D^* is $|D_{d+k}^0| = h_{d+k}$. From (6.21) and the Kruskal–Katona Theorem it follows that

$$h_{d+k-1} - h_{d+k} \geqq |\mathcal{S}_0| = f_{d-2}(D^*) \geqq f_{d-1}(D^*)^{(d)} = h_{d+k}^{(d)}.$$

This completes the proof of Theorem 6.1.

Remark. Let d be a fixed positive integer. For a graded left $\llcorner$-ideal M, define $f(M) = (f_{-1}(M), f_0(M), \dots)$, where $f_i(M) = \dim(M \cap \wedge^{i+1}V)$. Define $h(M)$ by equalities (1.5). If $H_{d+k}^{k+1}(M) = 0$ for all $k \geqq 0$, then $h(M)$ satisfies inequalities (1.7). The proof remains almost the same. The only difference is the replacement of $\bar{M}(C)$ by $\bar{M} = \{x \in \wedge V : \langle x, u\rangle = 0 \text{ for all } u \in M\}$.

7. Concluding remarks

(1) In [6] Eckhoff introduced the following four classes of simplicial complexes:

$\mathcal{K}^d$: the class of d-representable complexes.

$\mathcal{C}^d$: the class of d-collapsible complexes.

$\mathcal{H}^d$: $C \in \mathcal{H}^d$ iff $C = N(\mathcal{K})$, where $\mathcal{K}$ is a finite family of subsets of $\mathbf{R}^d$, and the intersection of every nonempty subfamily of $\mathcal{K}$ is either empty or a homology cell. ($C \subset \mathbf{R}^d$ is a homology cell if C is connected and $H_l(C)=0$ for $l=1,2,\ldots,d$.)

$\mathscr{L}^d$: the class of complexes C satisfying the following condition:

$$H_l(C/S)=0 \qquad \text{for every } l \geqq d \text{ and every } S \in C. \tag{7.1}$$

Obviously $\mathscr{K}^d \subset \mathscr{H}^d$. A theorem of Leray [14] implies that $\mathscr{H}^d \subseteq \mathscr{L}^d$. Wegner's Theorem asserts that $\mathscr{K}^d \subseteq \mathscr{C}^d$, and it is easy to see that $\mathscr{C}^d \subseteq \mathscr{L}^d$. (Wegner [20] conjectured that also $\mathscr{H}^d \subseteq \mathscr{C}^d$.) Eckhoff [6] conjectured that the sets of f-vectors of these four classes coincide, and are characterized by inequalities (1.7). In order to show that (1.7) holds for members of $\mathscr{L}^d$, it would suffice to prove the following conjecture (see Theorem 6.1):

Conjecture 7.1. *If $C \in \mathscr{L}^d$ then $H_{d+k}^{k+1}(C)=0$ for all $k \geqq 0$.*

(2) Eckhoff's h-conjecture resembles quite closely McMullen's g-conjecture concerning the characterization of f-vectors of simplicial d-polytopes (see [16, 17]). This conjecture was recently proved by Billera and Lee [1] (sufficiency) and Stanley [19] (necessity). Perhaps the methods developed here can be used to give an alternative proof for the necessity part of McMullen's conjecture.

(3) The generalized homology groups defined in Section 5 can be further generalized as follows: Let M be a graded left L-ideal of $\wedge V$.

Definition 7.2. For integers $r \geqq q \geqq 0$

$$Z_k^{r,q}=\left\{x \in M_k : \bigwedge^{q} F_r \,\llcorner\, x=0\right\} \tag{7.2}$$

$$(=\{x \in M_k : f_T \wedge x=0 \ \text{for all } T \in [r]^{(q)}\}),$$

$$B_k^{r,q}=\bigwedge^{r-q+1} F_r \,\llcorner\, M_{k+r-q+1}(\subset Z_k^{r,q}), \tag{7.3}$$

$$H_k^{r,q}(M)=Z_k^{r,q}/B_k^{r,q} \tag{7.4}$$

If C is a simplicial complex, define $H_k^{r,q}(C)=H_k^{r,q}(M(C))$.

Remark. $H_k^{r,r}(M)$ coincides with the group $H_k^r(M)$ defined in Section 5.

In the proof of the Katchalski–Perles conjecture in [11], we implicitly showed that if C is a d-collapsible complex and $\dim C=d+r$ then

$$H_{d+k}^{r+1,k+1}(C)=0 \qquad \text{for all } k \geqq 0. \tag{7.5}$$

(It is not hard to show that (6.1) already implies (7.5).) In fact, we proved in [11] that (7.5) holds even for "weakly (d, r)-collapsible complexes" (see [11, Sec. 5]). (For this wider class of complexes (6.1) need not hold.)

In [12] we define a graph G to be *k-acyclic* if $H_1^{k,1}(G) = 0$. We also define there the complementary notion of hyperconnectivity, and applied these notions to some extremal problems in graph theory.

Note added in proof. Conjecture 7.1 has now been proved.

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